ELSEVIER Journal of Pure and Applied Algebra 111 (1996) 21–29

JOURNAL OF PURE AND APPLIED ALGEBRA

An exterior product identity for Schur functions

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Communicated by J. Rhodes; received 15 April 1994; revised 5 April 1995

Abstract

Let A be a matrix in $\mathrm{Mat}_n(k)$, where k is a commutative ring. Let $\wedge^n\mathrm{Mat}_n(k)$ be the nth exterior power of $\mathrm{Mat}_n(k)$ as an n^2-dimensional free k-module. We present a coordinate-free characterisation of the Schur functions of (eigenvalues of) A, $s_\lambda(A)$, with $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$:

$$A^{\lambda_1+n-1} \wedge \cdots \wedge A^{\lambda_n+n-n} = s_\lambda(A) A^{n-1} \wedge \cdots \wedge A \wedge I.$$

This becomes the usual definition of the Schur functions when $A = \mathrm{diag}(x_1, \dots, x_n)$. A coordinate version of this identity was found earlier by A. Kilikauskas. We show how the "master identity" above may be used to derive new identities, and simplify the proofs of old identities involving Schur functions and linear recurrent sequences. We also discuss its place in algebra and Lie theory.

1. Exterior product identity

A Schur function is a symmetric polynomial which is a quotient of two alternating polynomials

$$s_\lambda(x_1, x_2, \dots, x_n) = \det(x_j^{\lambda_i+n-i}) / \det(x_j^{n-i})_{1 \le i,j \le n}, \tag{1}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a decreasing sequence of integers, $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$ [10].

If $x_1, \dots, x_n$ are eigenvalues of a generic $n \times n$ matrix A, one obtains a (scalar-valued) function, which is polynomial in the matrix elements and which we call a Schur function of A. We denote it $s_\lambda(A)$, with $s_\lambda(\mathrm{diag}(x_1, \dots, x_n))$ given by (1).

A. Kilikauskas [8] found that $s_\lambda(A)$ satisfies the following identity:

$$s_\lambda(A) = \det((A^{\lambda_i+n-i})_{jj}) / \det((A^{n-i})_{jj})_{1 \le i,j \le n}. \tag{2}$$

SSDI 0022-4049(95)00109-3

The purpose of this note is to prove the exterior product identity (3) and to show that (2), its generalisation (5), Jacobi–Trudi (8) and three more identities involving hook Schur functions (6), (7), (9) are its immediate consequences:

Theorem 1 (Exterior product identity (E.P.I)). *Let A be a generic matrix whose entries are commuting indeterminates a_{ij}, $1 \le i,j \le n$. The relation between exterior products,*

$$A^{\lambda_1+n-1} \wedge \cdots \wedge A^{\lambda_n+n-n} = s_\lambda(A) A^{n-1} \wedge \cdots \wedge A \wedge I \tag{3}$$

holds as an identity in the Grassmann algebra of the free n^2-dimensional module over $\mathbf{Z}[a_{ij}]$ generated by the matrices E_{kl} with only one nonzero entry.

The most obvious meaning of (3) is that the Schur function may be viewed as the determinant of the change of basis in the space of powers of a generic matrix A

$$(A^{n-1},\ldots,A,I) \longrightarrow (A^{\lambda_1+n-1},\ldots,A^{\lambda_n+n-n}). \tag{4}$$

Thus, $s_\lambda(A) = 0$ whenever $A^{\lambda_1+n-1},\ldots,A^{\lambda_n+n-n}$ are linearly dependent, except that the component of matrices satisfying $A^{n-1} \wedge \cdots \wedge A \wedge I = 0$ is removed.

Expanding (3) in the basis E_{kl} we get (2) with n arbitrary (distinct) pairs $k_j l_j$:

Theorem 2. *The Schur function $s_\lambda(A)$ satisfies the identity*

$$s_\lambda(A) = \det((A^{\lambda_i+n-i})_{k_j l_j}) / \det((A^{n-i})_{k_j l_j})_{1 \le i,j \le n}. \tag{5}$$

The identity (3) is related to the problem of expressing a general term of a linear recurring sequence as a linear combination of $n-1$ arbitrarily chosen terms. This problem was treated by W. Scheibner in 1864 (see [2,9] and Sections 2 and 4 below), and (3) could have been discovered at that time, but it was not. Interpreting hook Schur functions as solutions to a generic linear recurrence, we obtain three consequences of E.P.I.:

Theorem 3 (Hook Schur functions identity).

$$s_\lambda = (-1)^{\binom{n}{2}} \det(s_{(\lambda_i-i+1,1^{j-1})})_{1 \le i,j \le n}, \tag{6}$$

where $s_{(p,1^{j-1})} = (-1)^{j-1}\delta_{-p,j-1}$, if $-n+1 \le p \le 0$.

Theorem 4 (Kilikauskas' identity with hook Schur functions). *For $e_i = \operatorname{Tr} \wedge^i A$, and with the same convention on nonpositive arm lengths as in Theorem* 3,

$$s_\lambda = \det(s_{(\lambda_i-i+j,1^{n-j})})/e_1 e_2 \cdots e_{n-1}. \tag{7}$$

Theorem 5 (Jacobi–Trudi identity). *For $h_i = \operatorname{Tr} S^i A$ (where $S^i A : S^i(V) \to S^i(V)$)*

$$s_\lambda = \det(h_{\lambda_i-i+j})_{1 \le i,j \le n}. \tag{8}$$

Theorem 6. *For any $l \geq 0$, $k \geq 0$, $k \leq n$, the Schur functions $s_\lambda = s_{(\lambda',p+l,\lambda'')}$ ($\lambda' = (\lambda_1, \dots \lambda_{k-1})$, $\lambda'' = (\lambda_{k+1}, \dots, \lambda_n)$) satisfy the relation* [1]

$$s_{(\lambda',p+l,\lambda'')} = \sum_{j=1}^{n} (-1)^{j-1} s_{(p,1^{j-1})} s_{(\lambda',l+1-j,\lambda'')}. \tag{9}$$

In particular, for $k = n$ and/or for $p = 1$

$$s_{(\lambda',p+l)} = \sum_{j=1}^{n} (-1)^{j-1} s_{(p,1^{j-1})} s_{(\lambda',l+1-j)}, \quad s_{(\lambda',l+1)} = \sum_{j=1}^{n} (-1)^{j-1} e_j s_{(\lambda',l+1-j)}.$$

2. Proofs

Proof of Theorem 1. Since all polynomials in (3) have integer coefficients, it is sufficient to verify (3) on the (Zariski open) set of matrices with complex entries having no multiple eigenvalues, that is, for $A = BDB^{-1}$ with $D = \mathrm{diag}(x_1, \dots, x_n)$, $x_i \neq x_j$, $i \neq j$.

Let ρ be a representation of $GL(n)$ acting upon $\mathrm{Mat}_n(k)$ by conjugation: $\rho(B)(A) = BAB^{-1}$. $GL(n)$ acts on the exterior powers of the vector space $\mathrm{Mat}_n(k)$ by

$$\wedge^k \rho(B) : A_1 \wedge \cdots \wedge A_k \longrightarrow BA_1B^{-1} \wedge \cdots \wedge BA_kB^{-1}.$$

For $k = n$, i.e. for the nth exterior power of the n^2-dimensional vector space, $\mathrm{Mat}_n(k)$, and for any sequence of integers $p_1, \dots, p_n$, one has

$$A^{p_1} \wedge \cdots \wedge A^{p_n} = \wedge^n \rho(B)(D^{p_1} \wedge \cdots \wedge D^{p_n}). \tag{10}$$

Since the invertible operator $\wedge^n \rho(B)$ drops out from both sides of (3), this means that it is sufficient to prove (3) for diagonal matrices, in which case it reduces to definition (1) (the only nonzero coefficient is that of $E_{11} \wedge E_{22} \wedge \cdots \wedge E_{nn}$). $\square$

Proof of Theorem 2. Follows by expansion of (3) in the basis of exterior monomials $E_{k_1 l_1} \wedge \cdots \wedge E_{k_n l_n}$ for different sequences $(k_j l_j)_{1 \leq j \leq n}$. $\square$

Proof of Theorem 3. By Cayley's theorem, there is an expansion (cf. [6, 9])

$$A^{p+n-1} = \sum_{j=1}^{n} c_{p,j} A^{n-j}. \tag{11}$$

The coefficients $c_{p,j} = c_{p,j}(A)$ are obtained by Cramer's rule:

$$A^{p+n-1} \wedge A^{n-1} \wedge \cdots A^{n-j+1} \wedge A^{n-j-1} \wedge \cdots \wedge A \wedge I$$
$$= (-1)^{j-1} c_{p,j} A^{n-1} \wedge \cdots \wedge A \wedge I.$$

[1] If λ_i's are negative and/or not ordered, s_λ is *defined* by (3).

The powers on the left are $(p, 1^{j-1}, 0^{n-j}) + \delta$, where $\delta_i = n - i$. By (3),

$$\begin{aligned} c_{p,j} &= (-1)^{j-1} s_{(p,\,1^{j-1})}, \quad p > 0, \\ c_{p,j} &= \delta_{-p,j-1}, \quad -n+1 \le p \le 0. \end{aligned} \tag{12}$$

From now on, we define the hook Schur functions $s_{(p,1^{j-1})}$ for $-n+1 \le p \le 0$ as

$$s_{(p,1^{j-1})} = (-1)^{j-1}\delta_{-p,j-1}, \quad -n+1 \le p \le 0. \tag{13}$$

For $p = \lambda_1, \lambda_2 - 1, \ldots, \lambda_n - n + 1$, formula (12) yields

$$A^{\lambda_1+n-1} \wedge \cdots \wedge A^{\lambda_n+n-n} = \det((-1)^{j-1} s_{(\lambda_i-i+1,1^{j-1})}) A^{n-1} \wedge \cdots \wedge A \wedge I, \tag{14}$$

and the statement follows. As a byproduct we obtain a recursive expression for the power sums $\psi_p(A) = \operatorname{Tr} A^p$ via hook Schur functions:

$$\psi_{p+n-1} = \sum_{j=1}^{n} (-1)^{j-1} s_{(p,1^{j-1})} \psi_{n-j}. \qquad \square \tag{15}$$

Proof of Theorem 4. Let A be the matrix of a linear recurrence

$$u_{k+n} = a_1 u_{k+n-1} + \cdots + a_n u_k, \tag{16}$$

that is, a row of a's and a line of 1's under the main diagonal, e.g. for $n = 3$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The entries in the jth column of A^p are solutions to recurrence (16) with initial values in the jth column of the identity matrix:

$$(A^p)_{ij} = u^{(j)}_{p+n-j}, \quad \text{with} \quad u^{(j)}_{n-i} = \delta_{ij}, \ i = 1, \ldots, n, \tag{17}$$

the first row of A^p thus being

$$(A^p)_{1j} = u_{p+n-1} = \sum_{i=1}^{n} c_{p,i} u^{(j)}_{n-i} = \sum_{i=1}^{n} c_{p,i}\delta_{ij} = c_{p,j} = (-1)^{j-1} s_{(p,1^{j-1})}, \tag{18}$$

and (note that (13) is still in force)

$$(A^p)_{ij} = u^{(j)}_{p+n-i} = u^{(j)}_{(p-i+1)+n-1} = (-1)^{j-1} s_{(p-i+1,1^{j-1})}. \tag{19}$$

In particular, the last nonzero entry on the diagonal of A^p, $1 \le p \le n$, is $(A^p)_{pp} = (-1)^{p-1} s_{(1^p)} = (-1)^{p-1} e_p = a_p$, and all $(A^p)_{kk}$ with $k > p$ are equal to 0, since by our convention (13), for $i > p$, $s_{(p-i+1,1^{i-1})} = \delta_{i-p-1,i-1} = \delta_{0,-p}$. The denominator of (2),

$$\det\begin{pmatrix} * & \dots & a_1 & 1 \\ & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots \\ a_{n-1} & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 \end{pmatrix} = e_1 e_2 \cdots e_{n-1}.$$

(The sign of the permutation $i \to n-i$, $1 \le i \le n-1$, annihilates the signs coming from $a_i = (-1)^{i-1} e_i$.) The numerator of (2) is

$$(A^{\lambda_i+n-i})_{jj} = (-1)^{j-1} s_{(\lambda_i+n-i-j+1,1^{j-1})}$$

which after the substitution $j \mapsto n-j+1$ yields $\det(s_{(\lambda_i-i+j,1^{n-j})}) = s_\lambda \cdot e_1 \cdots e_{n-1}$. □

Proof of Theorem 5. By (19), $(A^p)_{j1} = (-1)^0 s_{(p-j+1,1^0)} = h_{p-j+1}$. For $p = \lambda_i + n - i$, coefficient of $E_{11} \wedge E_{21} \wedge \cdots \wedge E_{n1}$ in the expansion of $A^{\lambda_1+n-1} \wedge \cdots \wedge A^{\lambda_n+n-n}$ equals

$$\det(h_{(\lambda_i+n-i)-j+1}) = (-1)^{\binom{n}{2}} \det(h_{\lambda_i-i+j}).$$

(We changed $j \mapsto n-j+1$ as in the proof of Theorem 4.) In the expansion of $A^{n-1} \wedge \cdots \wedge A \wedge I$, this coefficient is a determinant of the form

$$\begin{vmatrix} * & 1 \\ 1 & 0 \end{vmatrix} = (-1)^{\binom{n}{2}},$$

and the result follows. □

Proof of Theorem 6. By (11), $A^{p+l+n} = \sum_{j=1}^{n} (-1)^{j-1} s_{(p,1^{j-1})} A^{l+n+1-j}$. Substituting A^{p+l+n} for A^{λ_k+n-k} in (3), with $\lambda_k + n - k = p + l + n$ (i.e. $\lambda_k = p + l + k$), we obtain

$$\begin{aligned} & s_{(\lambda',p+l+k,\lambda'')} A^{n-1} \wedge \cdots \wedge I \\ &= \sum_{j=1}^{n} (-1)^{j-1} s_{(p,1^{j-1})} A^{\lambda_1+n-1} \wedge \cdots A^{l+n+1-j} \cdots \wedge A^{\lambda_n} \\ &= \sum_{j=1}^{n} (-1)^{j-1} s_{(p,1^{j-1})} s_{(\lambda',l+k+1-j,\lambda'')} A^{n-1} \wedge \cdots \wedge A \wedge I, \end{aligned}$$

changing $l+k$ to l, and dividing by the discriminant $A^{n-1} \wedge \cdots \wedge I$ gives the result. □

3. Examples

This section contains low-dimensional examples which are intended to serve as an independent verification of the formulas developed thus far.

Recall that by the Jacobi–Trudi identity (see also the example for Theorem 6 below and (19) above)

$$\begin{aligned} s_{(2,1)} &= h_1 h_2 - h_3 = e_1 e_2 - e_3, \\ s_{(2,2)} &= h_2^2 - h_1 h_3 = e_2^2 - e_1 e_3. \end{aligned}$$

Example for Theorem 3 (identity (6)). For the shape  :

$$s_{(2,1)} = (-1)^{\binom{2}{2}} \begin{vmatrix} s_{(2-1+1,1^0)} & s_{(2-1+1,1^1)} \\ s_{(1-2+1,1^0)} & s_{(1-2+1,1^1)} \end{vmatrix} = - \begin{vmatrix} s_{(2)} & s_{(2,1)} \\ s_{(0,0)} & s_{(0,1)} \end{vmatrix}$$
$$= s_{(2,1)}s_{(0)} = s_{(2,1)}$$

(we used (13): $s_{(0,0)} = 1, s_{(0,1)} = 0$). For the shape ,

$$s_{(2,2)} = - \begin{vmatrix} s_{(2-1+1,1^0)} & s_{(2-1+1,1^1)} \\ s_{(2-2+1,1^0)} & s_{(2-2+1,1^1)} \end{vmatrix} = - \begin{vmatrix} s_{(2)} & s_{(2,1)} \\ s_{(1)} & s_{(1,1)} \end{vmatrix}$$
$$= s_{(2,1)}s_{(1)} - s_{(2)}s_{(1,1)}.$$

We check this result using the Pieri's rule, $s_\lambda s_{(m)} = \sum s_\nu$, ν being any partition "obtained by adding m boxes to the rows of λ, with no two boxes in one column" [5, p.455]:

$$s_{(2,1)}s_{(1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1^2)},$$
$$s_{(1,1)}s_{(2)} = s_{(3,1)} + s_{(2,1^2)},$$

and subtraction gives $s_{(2,2)}$.

Example for Theorem 4 (Kilikauskas' hook Schur functions identity):

$$s_{(2,1)}e_1 = \begin{vmatrix} s_{(2-1+1,1^{2-1})} & s_{(2-1+2,1^{2-2})} \\ s_{(1-2+1,1^{2-1})} & s_{(1-2+2,1^{2-2})} \end{vmatrix} = - \begin{vmatrix} s_{(2,1)} & s_{(3,0)} \\ s_{(0,1)} & s_{(1,0)} \end{vmatrix} = s_{(2,1)}s_{(1)}$$

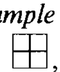
as required (since $s_{(1)} = e_1$). For $s_{(2,2)}$ we obtain

$$s_{(2,2)}e_1 = \begin{vmatrix} s_{(2-1+1,1^{2-1})} & s_{(2-1+2,1^{2-2})} \\ s_{(2-2+1,1^{2-1})} & s_{(2-2+2,1^{2-2})} \end{vmatrix} = - \begin{vmatrix} s_{(2,1)} & s_{(3,0)} \\ s_{(1,1)} & s_{(2,0)} \end{vmatrix}$$

which by Pieri's rule is equal to

$$(s_{(4,1)} + s_{(3,2)} + s_{(3,1,1)} + s_{(2,2,1)}) - (s_{(4,1)} + s_{(3,1,1)}) = s_{(3,2)} + s_{(2,2,1)}.$$

On the other hand, by the same rule $s_{(2,2)}s_{(1)} = s_{(3,2)} + s_{(2,2,1)}$, we are done.

Example for Theorem 6. We want to check that for $s_{(2,2)}$ with $n = 2$, i.e. for the shape ,

$$s_{(2,2)} = e_1 s_{(1,2)} - e_2 s_{(0,2)}.$$

Note that we used $k = 1$ (not $k = n$). (**Attention:** Subscripts $(1,2)$ and $(2,1)$ (or $(0,2)$ and $(2,0)$) refer to different Schur functions). By the Jacobi–Trudi identity for e_i [5, 10],

$$s_{(2,2)} = \begin{vmatrix} e_{2-1+1} & e_{2-1+2} \\ e_{2-2+1} & e_{2-2+2} \end{vmatrix} = \begin{vmatrix} e_2 & e_3 \\ e_1 & e_2 \end{vmatrix} = e_2^2,$$

since $e_3 = 0$ for $n = 2$. Now we calculate $s_{(1,2)}$ and $s_{(0,2)}$ by the E.P.I.:

$$s_{(1,2)}A \wedge I = A^{1+1} \wedge A^2 = 0, \qquad s_{(0,2)}A \wedge I = A^1 \wedge A^2 = -A^2 \wedge A^1 = -e_2,$$

and therefore $s_{(2,2)} = e_1 \cdot 0 - e_2 \cdot (-e_2) = e_2^2$, as required.

4. Discussion

Here we discuss the place of the exterior product identity in algebra and Lie theory.

1. Nice properties of the E.P.I. which show up here and there suggest that we should view it not as an *identity*, but rather as the right *definition* of Schur functions. Another option is to treat it as a current version of the Jacobi–Trudi identity ("Jacobi∧Trudi").

2. The polynomials $c_{p,j}$ (i.e. the hook Schur functions $(-1)^{j-1}s_{(p,\,1^{j-1})}$) arise in various settings. Fulton and Lang [6] use identities with $c_{p,j}$ in conjunction with "Grothendieck Riemann–Roch" theorem [6, II,III]. In [9], I applied these polynomials to the evaluation of linear recurrences on p processors. If one denotes $a(X) = X^n - \sum_{j=1}^n a_j X^{n-j}$, $c(X) = X^{p+n-1} - \sum_{j=1}^n c_{p,j}X^{n-j}$, and $\tilde{f}(X) = X^{\deg f} f(X^{-1})$, then

$$\tilde{c}(X) = (\tilde{a}(X)^{-1} \bmod X^p)\, \tilde{a}(X). \tag{20}$$

Since $\tilde{a}(X)^{-1} = \sum_{i=0}^{\infty} h_j X^j$ this yields an expression of $s_{(k,1^{j-1})}$ in terms of e_j and h_j:

$$s_{(p,1^{j-1})} = \sum_{k=0}^{p+j-1} (-1)^{j+k} e_k h_{p+j-1-k}. \tag{21}$$

3. Formula (3) makes sense for *any* sequence of integers $\lambda_1, \ldots, \lambda_n$, even for negative and not ordered λ's; $c_{p,j}$ with negative p may be calculated from recurrence (16), reverted:

$$u_k = -(a_{n-1}/a_n)u_{k+1} - (a_{n-2}/a_n)u_{k+2} - \cdots - (a_1/a_n)u_{k+n-1} + (1/a_n)u_{k+n},$$

resulting in, e.g., $s_{(-n,1^{j-1})} = u_{-1} = (-1)^j a_{j-1}/a_n = (-1)^{n-1}e_{j-1}/e_n$. Interpretation:

$$s_{(-n,1^{j-1})} = (-1)^{n-1}\mathrm{Tr} \wedge^{n-j+1} A^{*-1} = (-1)^{n-1}\mathrm{Tr} \wedge^{n-j+1} \rho^*(A),$$

where ρ is a standard representation of $GL(n)$.

4. Theorem 5 admits the following interpretation. Let the exterior monomials $A_1 \wedge \cdots \wedge A_m$ act on $V^{\otimes m}$ as antisymmetrized products

$$\sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{\sigma 1} \otimes \cdots \otimes A_{\sigma m}.$$

For $v_0 = (1, 0, \ldots, 0)$, the first column of A^p contains recurrence entries $u_k = h_k(A)$,

$$A^p(v_0) = (h_p, h_{p-1}, \ldots, h_{p-n+1})$$

and the exterior product of A's assumes on $v_0^{\otimes n}$ the value

$$A^{\lambda_1+n-1} \wedge \cdots \wedge A^{\lambda_n+n-n}(v_0 \otimes \cdots \otimes v_0) = \det(h_{\lambda_i-i+j})_{1\le i,j\le n}\ \ v_{n-1} \wedge \cdots \wedge v_0,$$

where v_k is a vector consisting of k zeros, one 1 and $n-k-1$ zeros.

5. Since the formulation of E.P.I. is functorial, it may be conjectured that it holds in a wider Lie-theoretic context (e.g. for super or quantum versions of $GL(n)$ [12]). It also seems plausible that the Weyl character formula [5, 15] is related to E.P.I.

6. Invariants for orthogonal and symplectic Lie algebras may be constructed using linear relations between even or odd powers of A, e.g. for $2n \times 2n$ matrices

$$A^{2(\lambda_1-n+1)} \wedge \cdots \wedge A^{2\lambda_n} = f(A)A^{2(n-1)} \wedge \cdots \wedge A^2 \wedge I.$$

Invariants of m matrices (cf. [3]) $A_1,\dots,A_m$ may be obtained by picking $m(n-1)+1$ factors from mn terms $A_j^{\lambda_i-n+i}$ and expressing their exterior product as $f(A_1,\dots,A_m)$ $A_1^{n-1} \wedge \cdots \wedge A_1 \wedge A_2^{n-1} \wedge \cdots \wedge A_2 \cdots A_m^{n-1} \cdots A_m \wedge I$. It is not clear though whether these systems of invariants will be complete.

7. A set-theoretic pattern beyond Schur functions is as follows: one has a set S with two maps, $F : S \longrightarrow S$, and $W : S \times \cdots \times S \longrightarrow T$ (n-fold product of S to T). A Schur function $s_{\lambda,\mu}(F)$ is then a morphism $s_{\lambda,\mu} : T \longrightarrow T$, which transforms a map $W \circ F^{\mu-\delta} = W(F^{\mu_1+n-1}(s),\dots,F^{\mu_n+n-n}(s)) : S \longrightarrow T$ to the map $W \circ F^{\lambda-\delta}$:

$$\begin{tikzcd} & S \arrow[dl, "W \circ F^{\mu+\delta}"'] \arrow[dr, "W \circ F^{\lambda+\delta}"] & \\ T \arrow[rr, "s_{\lambda\mu}(F)"] & & T \end{tikzcd}$$

(for $\mu = 0$, $s_{\lambda\mu}(F)$ is what we denoted by $s_\lambda(F)$).

8. Coefficients relating terms of linear recurrent sequences (cf. [2]),

$$u_k = d_1 u_{\lambda_1+n-1} + \cdots + d_n u_{\lambda_n+n-n}$$

are easily expressed in terms of Schur functions (Cramér's rule again):

$$d_i = s_{(\lambda_1,\dots,\lambda_{i-1},k-n+i,\lambda_{i+1},\dots,\lambda_n)}/s_\lambda.$$

9. It would be very interesting to find a version of E.P.I for Schur-like functions with the Wronskian [4, 14] or Muir determinant [1, 7, 11] in the denominator (cf. (1)).

10. It is hard to believe that the identities with hook Schur functions (Theorems 3 and 4 above) were really unknown. Apparently, they were treated as consequences of the Jacobi–Trudi identity, from which they may be derived by row operations [13].

Acknowledgements

Many thanks to Prof. Jeffrey Remmel for the introduction to the results of his Ph.D. student A. Kilikauskas, for many helpful discussions, and for the statement of Theorem 4. I greatly benefited from discussions of this result with Prof. Nolan Wallach. Thanks to Prof. Ian G. Macdonald who assured me that E.P.I. has not attracted his attention before. I am indebted to Prof. Jeffrey Rabin, Noam Elkies and David Goss for discussions and references, and to William Brockman for his generous linguistic help. I am sincerely thankful to Prof. Adriano Garsia, Dr. Ordjonika Koutina and Prof. Arkady Vaintrob for their help and encouragement.

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